Physics 200A Fall 2016

I.) Review of Basic Lagrangian Mechanics (See L/L Chapt 1, 2)

→ Principle of Least Action / Hamilton's Principle (— Physics from a Variational Principle).

If given a system of <u>point particles</u> (i.e. now internal d-o-fs, <u>yet</u>), such that:

- parametrized by <u>generalized coordinates</u> $q_1, q_2 \ldots q_s$ and <u>generalized velocities</u> $\dot{q}_1, \dot{q}_2 \ldots \dot{q}_s$ where $q_i = q_i(t)$.

(n.b. time is <u>parameter</u>).

Note: Generalized coordinate need <u>not</u> correspond to standard coordinates → essence of Lagrangian approach, very useful for problem solving.

- let Lagrangian be described by a function $L(q_i, \dot{q}_i, t)$

  explicit time dependance is possible

then trajectory $q_1(t_1) \rightarrow q_2(t_2)$ is one which minimizes action S

$$\underbrace{S}_{\text{action}} = \int_{t_1}^{t_2} dt\, L(q_i, \dot{q}_i, t)$$

- function of trajectory $q_i(t)$

- minimization enables g.c.

trajectory selected by PLA.

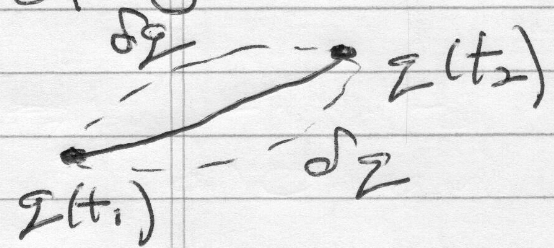
Some observations / comments:

- variational principle allows use of generalized coordinates. S is minimized, for any parametrization.

- S not calculated. EOM are. Hamilton-Jacobi theory calculates S.

- PLA is energy method, as $L = T - U$ (tbd).

Now, consider extremization/variation of S:



$$\delta S = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q}\,\delta q + \frac{\partial L}{\partial \dot{q}}\,\delta \dot{q} \right)$$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial \dot{q}}\,\frac{d}{dt}(\delta q) + \frac{\partial L}{\partial q}\,\delta q \right)$$

chop

$$= \frac{\partial L}{\partial \dot{q}}\,\delta q \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\, \delta q(t) \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right)$$

but $\delta q(t_{1,2}) = 0$ no variation of end points

so

$$\delta S = \int_{t_1}^{t_2} dt \; \delta q(t) \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right)$$

and $\delta S =$ for <u>all</u> δq <u>if</u>:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} = 0} \rightarrow \text{Lagrange's Equation}$$

- LEOM determine $q(t)$ trajectory. Need i.c. for solution

- LEOM are PDE

- Lagrangian – and thus all physics – is invariant under addition of df/dt

can have expl. dep.

i.e. $$S = \int_{t_1}^{t_2} dt\, L = \int_{t_1}^{t_2} dt \left(L + df/dt\right)$$
$$= S_0 + f(q(t_2)) - f(q(t_1))$$

<u>oo</u>

$\delta S = \delta S_0 + \delta(\Delta f)$

but $\delta q(t_{2,1}) = 0$

i.e. $\delta \Delta f = \Delta \frac{\partial f}{\partial q} \delta q \to 0$

so $\delta S = \delta S_0$ → no change in trajectory or physics.

Some obvious questions:

- is $q(t)$ obtained as a minimum of S.
  - only assured an extremum (stationary point). In practice, a minimum.
  - meaning from L.

- what is L?

but 'we know $L = T - V$'

kinetic energy

potential energy.

but what if we did not ...

⇒ structure of L:

- consider approach by symmetry

[redacted]

- start with free particle

→ non-relativistic

→ space-time homogeneity ⇒
L indep. $\underline{x}, t$

$$L = L(\underline{v}) \text{ only}$$

→ space-time isotropy ⇒ $L =$
$L(v^2) = L(\underline{v} \cdot \underline{v})$
($\underline{v}$ → direction dependence)

⇒ Non-relativistic physics
must be invariant to
Galilean boost

i.e. ⇒

Principle of (Galilean) Relativity:

For two frames of reference related
by infinitesimal Galilean boost,

trajectories must be the same.

i.e. boost:
$$\underline{r} = \underline{r}' + \underline{V}\, t$$
$$\underline{v} = \underline{v}' + \underline{V}$$
$$t = t'$$

Approach: Show $L((\underline{v}+\delta\underline{v})^2)$ and $L(v^2)$ differ by df/dt if $\partial L/\partial v^2 = \text{const}$.

Check:
$$L\left[(\underline{v}+\delta\underline{v})^2\right] - L(v^2)$$
$$\cong \cancel{L(v^2)} + (2\,\underline{v}\cdot\delta\underline{v})\left(\frac{\partial L}{\partial v^2}\right) + \delta v^2\left(\frac{\partial L}{\partial v^2}\right) - \cancel{L(v^2)}$$
$$\cong (2\,\underline{v}\cdot\delta\underline{v})\,\frac{\partial L}{\partial v^2}$$

Now, if $\partial L/\partial v^2 = \text{const.}$ (i.e. indep v^2), then

$$L\left((\underline{v}+\delta\underline{v})^2\right) - L(v^2) \cong m\,\underline{v}\cdot\delta\underline{v}$$

$\delta\underline{v}$ is parameter

$$= \frac{d}{dt}\left(m\,\underline{x}\cdot\delta\underline{v}\right)$$

$$= \frac{df}{dt} \rightarrow \text{so irrelevant.}$$

$\therefore$

- $\partial L/\partial v^2 = \text{const} = m/2$, for correspondence.

- L Galilean invariant

- $m > 0$ for <u>minimum</u> in S.

Thus, for free particle:

$L = (m/2)\,v^2$ $\Rightarrow$ from symmetry and Galilean relativity

$\Rightarrow$ Newton's first Law.

N. B.: Show! (Trajectory free particle unchanged)

$$\Delta L = \frac{m}{2}(\underline{v}+\underline{V})^2 - mv^2/2$$

$$= \cancel{\frac{m}{2}v^2} + m\underline{v}\cdot\underline{V} + \frac{m}{2}\underline{V}^2 - \cancel{\frac{mv^2}{2}}$$

$$= \frac{d}{dt}\left(m\underline{x}\cdot\underline{V} + \frac{1}{2}m\underline{V}^2 t\right) \checkmark$$

so

$$L_{\text{free particle}} = mv^2/2$$

for standard coordinates

$$T = 1/2\, m\,(dl/dt)^2$$

$$dl^2 = dx^2 + dy^2 + dz^2 \quad \text{(Cartesian)}$$

$$dl^2 = dr^2 + r^2 d\theta^2 + dz^2 \quad \text{(Cylinder)}$$

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

(Sphere)

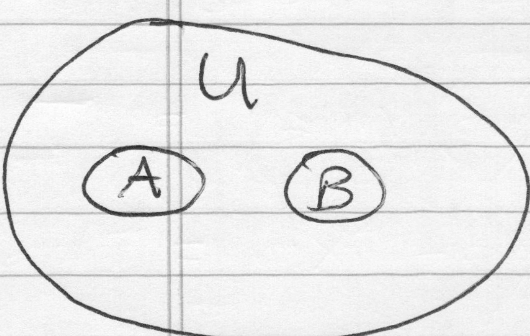
$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$T = \frac{m}{2} (\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$$

$$T = \frac{m}{2} (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2)$$

For interacting particles, i.e. not free:

- useful to introduce concept of open, closed system



$U \equiv$ universe

A, B $\rightarrow$ systems

systems:

closed $\rightarrow$ non-interacting

open $\rightarrow$ interacting

if $\mathcal{U}$ formed by two closed subsystems A, B (i.e. two non-interacting free particles)

$$L_{\mathcal{U}} = L_A + L_B$$

i.e. Lagrangians for closed sub-systems additive ⇒ i.e. L for system must asymptote to that for sum of L's for individual sub-system, at large separation.

[consider 2 particles ⇒ must go to two individual free particles].

Now, in non-relativistic limit:

For system of interacting particles which is closed, Lagrangian can be written as:

$$L = \sum_i \frac{m_i v_i^2}{2} + \underbrace{\varphi(\underline{r}_1, \underline{r}_2 \ldots)}$$

interaction potential

⇔ function of coordinates only.

n.b.: – $v \ll c$ $(c \to \infty)$

so

– $r\left(t - \frac{|\Delta r|}{c}\right) \to r(t)$
(distance)

"really, particle "feels" effect of neighbor at retarded time

i.e. $\frac{dp_1}{dt} - \frac{\partial \varphi}{\partial r_1}(r_1, r_2, t) = 0$

$$r_2 = r_2\left(t - \frac{|r_1 - r_2|}{c}\right) \xrightarrow[c\to\infty]{} r_2(t)$$

→ Now, in event that $z_i = x_i$ (g.c.'s are Cartesian coords), know LEOMs must reduce to Newton's Laws

(Correspondence argument for φ, L)

L.E: $\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) - \frac{\partial L}{\partial x_i} = 0$

so

$$\frac{d}{dt}(p_i) - \frac{\partial Q}{\partial x_i} = 0$$

so $Q = -V = -U$

Newtonian potential

so

$$\boxed{L = T - U}$$

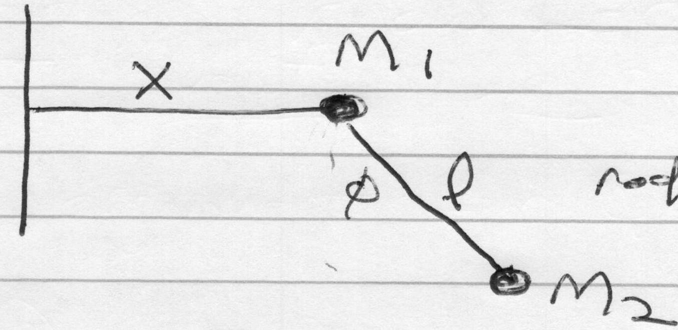
terminology: - generalized coordinate q_i

- generalized momentum $\partial L / \partial \dot{q}_i = p_i$

Examples

i.) (Trivial)

Pendulum attached to freely sliding mass



Lagrangian and LEOMs.

G.C.: ϕ, x

m_1: $T_1 = \frac{1}{2} m_1 \dot{x}^2$

$U = 0$

m_2: $T_2 = \frac{1}{2} m_2 (\dot{x}^2 + \dot{y}^2)$

$= \frac{1}{2} m_2 \left[(\dot{x} + l \dot{\sin\phi})^2 + (l \dot{\cos\phi})^2 \right]$

$U = mgl(1 - \cos\phi)$

$\dot{l} = 0$

so

$$L = \frac{m_1}{2} \dot{x}^2 + \frac{m_2}{2} \left[\dot{x}^2 + 2\dot{x} l \dot{\phi} \cos\phi + l^2 \cos^2\phi \, \dot{\phi}^2 + l^2 \sin^2\phi \, \dot{\phi}^2 \right] - mgl(1 - \cos\phi)$$

$$= \frac{(m_1 + m_2)}{2} \dot{x}^2 + m_2 \left[\frac{l^2}{2} \dot{\phi}^2 + \dot{x} l \dot{\phi} \cos\phi \right] + mgl\cos\phi - \cancel{mgl}$$

LEOM: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}}\right) - \dfrac{\partial L}{\partial x} = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

1) $(m_1 + m_2)\ddot{x} - m_2 l \ddot{\phi}\cos\phi = 0$

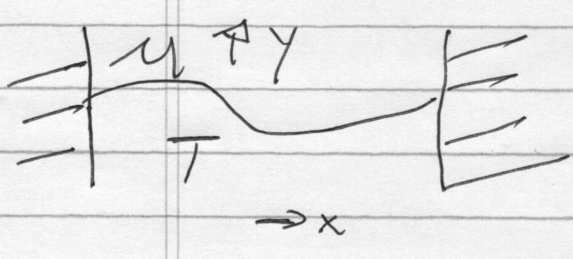
$$\frac{d}{dt}\left((m_1+m_2)\dot{x} - m_2 l \sin\phi\right) = 0$$

(conservation of $\vec{x}$ mom.)

2) $\dfrac{d}{dt}\left(m_2 l^2 \dot{\phi} + m_2 l \cos\phi\right) + mgl\sin\phi = 0$

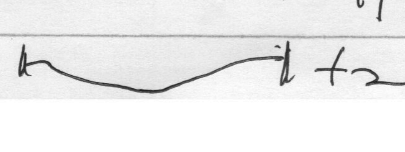
ii.) <u>Non-trivial</u> (something of a preview)

Equation for 1D nonlinear string with tension T, mass-per-length μ!



- end-points fixed $\delta y(x=0) = \delta y(L) = 0$, all t
- $\delta y(x, t_2, t_1) = 0$

G.C. = $y(x,t)$ — x a param.



known initial, final conformation

$$\sum_i = \int dx$$ $\rightarrow$ x, like t is a parameter.

so

$$S = \int_{t_1}^{t_2} dt \, L$$

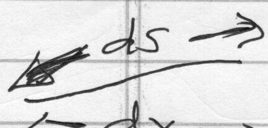
$$= \int_{t_1}^{t_2} dt \int_0^L dx \, \mathcal{L}$$

$\downarrow$ Lagrangian density

$$U = \int_0^L ds \, T - \int_0^L dx \, T$$

$\downarrow$ net potential energy stored by 'plucking' string

$\downarrow$ unperturbed energy



$$U = \int_0^L dx \left(\frac{ds}{dx} - 1 \right) T$$

$$ds^2 = dx^2 + dy^2$$

$$= dx^2\left(1 + (dy/dx)^2\right)$$

so (n.b. x, t indep. parameters):

$$\mathcal{L} = \tfrac{1}{2}\mu\left(\partial_t y(x,t)\right)^2 - T\left(1 + (\partial y/\partial x)^2\right)^{1/2}$$

$$S = \int_{t_1}^{t_2} dt \int_0^L dx\, \mathcal{L}$$

$$\delta S = \int_{t_1}^{t_2} dt \int_0^L dx \left\{ \frac{\partial \mathcal{L}}{\partial y_t}\,\delta y_t + \frac{\partial \mathcal{L}}{\partial y_x}\,\delta y_x \right\}$$

$$= \cancel{\int_0^L dx \frac{\partial \mathcal{L}}{\partial y_t}\,\delta y \Bigg|_{t_1}^{t_2}} + \underbrace{\cancel{\int_{t_1}^{t_2} dt \frac{\partial \mathcal{L}}{\partial y_x}\,\delta y \Bigg|_0^L}}_{0}$$

initial final config. fixed.

e.p. fixed

$$- \int_{t_1}^{t_2} dt \int_0^L \left[\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) \right]$$

so $\delta S = 0 \Rightarrow$

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) = 0}$$

$$\frac{\partial \mathcal{L}}{\partial y_t} = \mu y_t \,,$$

$$\frac{\partial \mathcal{L}}{\partial y_x} = \frac{-T\, y_x}{\left[1 + y_x^2\right]^{1/2}}$$

so

$$\boxed{\partial_t\, \mu \frac{\partial y}{\partial t} - \frac{\partial}{\partial x}\left(\frac{T\, \partial y/\partial x}{\left(1 + \left(\partial y/\partial x\right)^2\right)^{1/2}}\right) = 0}$$

NL equation for string (clamped).

- Linearizing:

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}$$ Wave Eqn.